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MINIMUM-WEIGHT DESIGN OF HOLLOW CYLINDERS FOR GIVEN LOWER BOUNDS ON TORSIONAL AND FLEXURAL RIGIDITIES

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Abstract—An approximate analytical solution is obtained for the problem of minimizing the cross-sectional area of elastic, doubly-connected cylindrical bars for given lower bounds on torsional and flexural rigidities. The bars are designed to withstand either a twisting or a bending moment, but not both, at any given time. The shape of the inner contour of the doubly-connected (hollow) cross-section is specified while that of the outer contour is determined as a result of the optimization. The economy achieved by optimization is estimated.

INTRODUCTION

An earlier paper by one of the authors[1] dealt with the problem of minimizing the cross-sectional area (the weight) of elastic cylindrical bars with singly-connected (solid) cross-section of given torsional and bending stiffnesses. This work is extended in the present paper to include doubly-connected (hollow) cross-sections. The extension, however, is by no means straight forward. Whereas the former problem lent itself to a closed-form analytical solution, no such solution seems to be feasible to the problem under study. Therefore, an approximate (analytical) solution is presented. The solution technique is similar to that used in [2] for finding the cross-sectional shape of hollow elastic bars that have the maximum torsional stiffness.

The cylindrical bar with a doubly-connected (hollow) cross-section, the area of which is being minimized, is expected to withstand either a twisting or a bending moment, but not both, at any given time. Moreover, the shape of the inner contour is assumed to be known, while that of the outer contour is to be determined such that the cross-section has given torsional and bending stiffnesses.

Such multi-purpose optimal design problems are of both theoretical and practical interest and have been considered recently in [1, 3]. These problems differ from those studied earlier[4, 5] involving multiple loadings of deterministic or stochastic type[6], in that, besides an increase in the number of design parameters, they are no longer one-dimensional but are described by partial differential equations, the partial derivatives appearing both in the differential equation and the necessary optimality condition.

Thus, in Section 1 we give a mathematical statement of the problem and of the necessary optimality condition. Section 2 deals with the solution of the problem using a perturbation technique. The latter allows us to obtain approximate analytical solutions with any desired degree of accuracy. However, it is shown that, already in the first approximation, sufficient economy is achieved by optimization, and any further refinement in the solution would not be worth the effort involved. The first approximation more or less reveals the optimal cross-sectional shape. Moreover, for a better appreciation of the effectiveness of optimization, the inner contour of the hollow cross-section is assumed to be circular in shape, although the statement of the problem and the corresponding optimality condition are applicable to any arbitrary shape. Finally, in Section 3 we present the results graphically as a function of the given design parameters—torsional and bending stiffnesses—and discuss them in the light of the economy possible due to optimization.

1. STATEMENT OF THE PROBLEM AND THE NECESSARY OPTIMALITY CONDITION

As noted above, the cylindrical bar under study is to withstand either a twisting or a bending moment, but not both, at any given time. In other words, during its design life it will act at times as a shaft and at other times as a beam. It is thus necessary to consider one by one its design purposes.

Consider a homogeneous isotropic cylindrical bar with a doubly connected (hollow) cross section that is required to transmit twisting moment M applied at its ends, the torsional stiffness J, the angle of twist per unit length of the bar θ and M are related through the well known formula $M = \mu J\theta$, where μ is the shear modulus. Let the solid part of the cross-section be designated by D (Fig. 1).

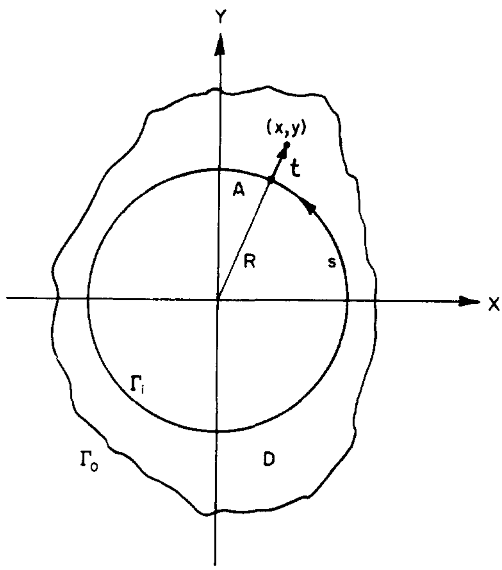

Fig. 1

Furthermore, let the contour Γ_0 bound this region from without, and the contour Γ_i from within.

The non-zero components of the stress tensor τ_{xz} and τ_{yz} (z-axis is directed along the length of the bar) can be expressed through a stress function $\phi(x, y)$

$$\tau_{xz} = \mu\theta\phi_y,\ \tau_{yz} = -\mu\theta\phi_x \tag{1}$$

which satisfies Poisson's equation (see, for example,[7])

$$\phi_{xx} + \phi_{yy} = -2 \tag{2}$$

and the boundary condtions

$$\phi = 0, (x, y) \in \Gamma_0 \tag{3}$$

$$\phi = C,\ (x, y) \in \Gamma_i. \tag{4}$$

Here $\phi_x = \partial\phi/\partial x$, $\phi_{xx} = \partial^2\phi/\partial x^2$, and so on, and C is an unknown constant related to ϕ (follows from Bredt's theorem) through

$$\int_{\Gamma_i} \frac{\partial\phi}{\partial n}\,\mathrm{d}s = -2\Omega \tag{5}$$

where Ω is the area of the region bounded by Γ_i and n, as usual, is the inward normal to Γ_i.

The torsional stiffness of a hollow cross-section is given by

$$J = 2\left(\iint_D \phi \,\mathrm{d}x\,\mathrm{d}y + C\Omega\right). \tag{6}$$

The design constraint has the form

$$J \ge J^0. \tag{7}$$

Next, consider the function of the bar as a beam. The basic mechanical property both from the flexure and vibration point of view is the bending stiffness I. If the bending takes place in the yz-plane, then the design constraint may be expressed as

$$I = \iint_D y^2 \,\mathrm{d}x\,\mathrm{d}y \ge I^0. \tag{8}$$

If the shape of the boundary Γ_i is assumed to be known the optimization problem under study reduces to determining the shape of the boundary Γ_0 that fulfils the design requirements (7)–(8) and minimizes the area of the region D

$$S(\Gamma_0) = \iint_D \mathrm{d}x\,\mathrm{d}y \to \min \tag{9}$$

To derive the necessary optimality condition, it is expedient to eliminate the differential constraint (2). To this end, we utilize the well-known result[7] that, if the boundary contour Γ_0 is specified, the function ϕ which minimizes the functional

$$J = \min_\phi \frac{1}{2}\iint_D (\phi_x^2 + \phi_y^2 - 4\phi)\,\mathrm{d}x\,\mathrm{d}y - 4C\Omega \ge J^0 \tag{10}$$

is a solution of the boundary-value problem (2)–(4), where the minimum with respect to ϕ is sought among a class of continuously differentiable functions that satisfy the boundary conditions (3)–(4) and the condition (5). However, it can be shown that (5) is a "natural" condition for the functional (10) and will thus be automatically satisfied by the function which minimizes (10) subject only to the boundary conditions (3)–(4).

To derive the necessary optimality condition for the problem (7)–(10), we employ the Lagrange multiplier technique and construct an auxiliary functional

$$\Pi = \iint_D \mathrm{d}x\,\mathrm{d}y + \lambda_1\left\{\frac{1}{2}\iint_D (\phi_x^2 + \phi_y^2 - 4\phi)\,\mathrm{d}x\,\mathrm{d}y - 4C\Omega - J^0\right\} + \lambda_2\left\{\iint_D y^2\,\mathrm{d}x\,\mathrm{d}y - I^0\right\} \tag{11}$$

and write an expression for its first variation

$$\delta\Pi = \int_{\Gamma_0} \delta f\,\mathrm{d}s + \frac{\lambda_1}{2}\int_{\Gamma_0} (\phi_x^2 + \phi_y^2)\,\delta f\,\mathrm{d}s + \lambda_2\int_{\Gamma_0} y^2\delta f\,\mathrm{d}s, \tag{12}$$

where δf denotes the normal displacement of points on the unknown boundary Γ_0 due to a variation of the region D. In writing the expression for the variation of the functional, $\delta\Pi$, we utilized the eqns (2)–(5) and the fact that $\delta\phi$ is arbitrary. As δf is also arbitrary, the stationarity property $\delta\Pi = 0$ gives the necessary optimality condition

$$\phi_x^2 + \phi_y^2 = \mu_1 + \mu_2 y^2, \tag{13}$$

where $\mu_1 = -2/\lambda_1$ and $\mu_2 = -2\lambda_2/\lambda_1$ are the new Lagrange multipliers to be determined from (7) and (8). The optimality condition which defines the outer contour Γ_0 could also be derived by introducing a new function that reduces the differential eqn (2) to Laplace's equation. The procedure is similar to that used in [2]. A comparison with the corresponding optimality condition for a bar with a solid cross-section[1] immediately reveals that the optimality condition is independent of whether or not the cross-section is singly or multiply-connected.

Thus the shape of the contour Γ_0 and the corresponding stress function $\phi(x, y)$ can be found from (2)–(4) and (13). However, as was noted above, this problem does not seem to have a closed-form solution. We present in the next section an approximate analytical solution technique similar to the one used in [2].

2. METHOD OF SOLUTION

The preceding discussion is applicable to any given shape of the inner boundary Γ_i. However, for a better appreciation of the perturbation technique used to solve the optimization problem under study we restrict our attention to a specific shape of Γ_i, namely circular with radius R.

As with the problem of optimizing the shape of a cylindrical bar with a solid cross-section[1], it is likely that there exist regions in the plane of design parameters (J, I) in which the optimal design is governed by only one of the two design parameters. In fact, it is easily shown that, if the given design contraints J^0 and I^0 satisfy the inequality

$$\frac{I^0}{J^0} \leq 1/2, \tag{14}$$

that is the design parameters lie in the region 1 (Fig. 2), the outer boundary of the optimal bar must be circular in shape, i.e. the same as the inner boundary. This follows from the fact that if the inequality (14) is observed by a hollow circular bar of torsional stiffness J^0, the bending stiffness of such a bar will be automatically greater than I^0.

The indicated division of the plane of design parameters (J, I) also follows from the solution of the problem (see below).

In using the perturbation technique described below it is convenient to change to an orthogonal curvilinear coordinate system (s, t), where s is the distance measured along the inner contour Γ_i (Fig. 1) from the horizontal axis (x-axis) normal to the plane of bending to a

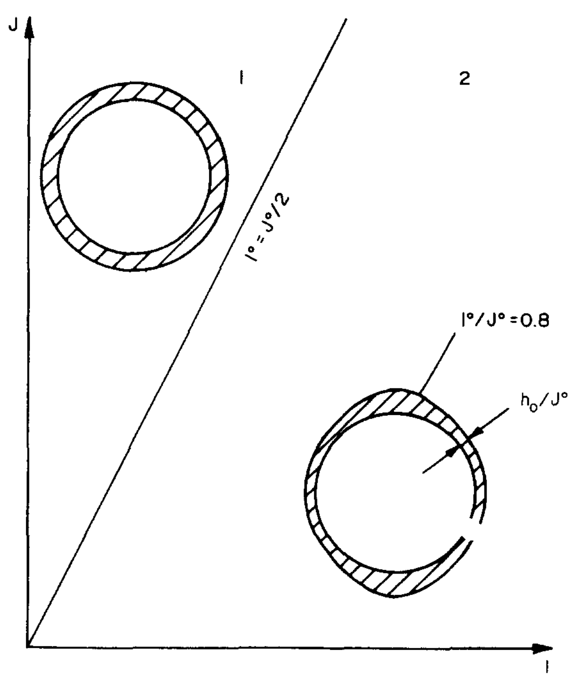

Fig. 2

point A on this contour and t is the distance along the normal to the contour Γ_i measured from A to a point (x, y) in the region D.

Let $L = 2\pi R$ denote the perimeter of the inner circular boundary Γ_i and $h = h(s)$ be the equation to the unknown outer boundary Γ_0. In the new coordinate system the differential eqn (2) and the boundary conditions (3)–(4) take the following form

$$T\phi_{tt} + \frac{1}{T}\phi_{ss} + \frac{1}{R}\phi_t = -2T, \tag{15}$$

$$\phi(s, h) = 0, \tag{16}$$

$$\phi(s, 0) = C, \tag{17}$$

where the variable T is given by $T = 1 + t/R$.

The eqn (5) defining the constant C takes the form

$$\int_0^L \phi_t(s, 0)\,\mathrm{d}s = -2\Omega, \tag{18}$$

and the design constraints the form

$$J = 2\left(\int_0^L \int_0^h T\phi\,\mathrm{d}t\,\mathrm{d}s + C\Omega\right) \geq J^0, \tag{19}$$

$$I = R^2 \int_0^L \int_0^h T^3 \sin^2\left(\frac{s}{R}\right)\mathrm{d}t\,\mathrm{d}s \geq I^0 \tag{20}$$

The optimization problem is thus reduced to determining the function $h(s)$ and the stress function $\phi(s, t)$ that satisfy the differential equation (15) and the boundary conditions (16)–(17) and minimize the functional (area of cross-section)

$$S = \int_0^L \int_0^h T\,\mathrm{d}t\,\mathrm{d}s \to \min \tag{21}$$

subject to the constraints (19)–(20).

The condition (13) necessary for achieving this optimal design has the following form

$$\phi_t^2 + \frac{1}{T^2}\phi_s^2 = \mu_3^2\left(1 + \mu_4 T^2 \sin^2\frac{s}{R}\right), \tag{22}$$

where $\mu_3^2 = \mu_1$ and $\mu_3^2\mu_4 = \mu_2$.

As the stress function vanishes along the outer boundary Γ_0, $t = h$, i.e. $\phi(s, t) = 0$, it follows that $\phi(s, h) = 0$, and the necessary optimality condition reduces to

$$\phi_t^2(s, h) = \mu_3^2\left[1 + \mu_4\left(1 + \frac{h}{R}\right)^2 \sin^2\frac{s}{R}\right]$$

or

$$\phi_t(s, h) = -\mu_3\sqrt{\left(1 + \mu_4\left(1 + \frac{h}{R}\right)^2 \sin^2\frac{s}{R}\right)}. \tag{23}$$

If μ_3 is assumed to be positive, the choice of negative sign in (23) is dictated by (18).

Henceforth, we assume the cross-section to be thin-walled such that $\max_s h(s) = H \ll L$ for $0 \leq s \leq L$ and $\epsilon = H/L$ is a small parameter.

Let us non-dimensionalize the system of eqns (15)–(23) by introducing the following variables

$$s = Ls',\ t = Ht',\ h = Hh',\ \phi = HL\phi',\ \Omega = L^2\Omega',\ S = HLS',\ R = LR',\ J = HL^3J',$$
$$I = HL^3I',\ \mu_3 = L\mu_3',\ C = HLC'. \tag{24}$$

In the new non-dimensional variables (15)–(23) take the following form (primes have been omitted)

$$T\phi_{tt}+\frac{\epsilon^2}{T}\phi_{ss}+\frac{\epsilon}{R}\phi_t=-2\epsilon T,\ T=1+\epsilon\frac{t}{R} \tag{25}$$

$$\phi(s,h)=0 \tag{26}$$

$$\phi(s,0)=C \tag{27}$$

$$\int_0^1 \phi_t(s,0)\,\mathrm{d}s=-2\Omega \tag{28}$$

$$J=2\left(\epsilon\int_0^1\int_0^h T\phi\,\mathrm{d}t\,\mathrm{d}s+C\Omega\right)\geq J^0 \tag{29}$$

$$I=R^2\int_0^1\int_0^h T^3\sin^2\left(\frac{s}{R}\right)\mathrm{d}t\,\mathrm{d}s\geq I^0 \tag{30}$$

$$S=\int_0^1\int_0^h T\,\mathrm{d}t\,\mathrm{d}s\to\min \tag{31}$$

$$\phi_t(s,h)=-\mu_3\sqrt{\left(1+\mu_4\left(1+\epsilon\frac{h}{R}\right)^2\sin^2\frac{s}{R}\right)}. \tag{32}$$

We seek a solution of the problem in a series of the small parameter ϵ. In this connection we expand all the variables in powers of ϵ,

$$\begin{aligned}\phi&=\phi_0+\epsilon\phi_1+\epsilon^2\phi_2+\ldots.\\ C&=C_0+\epsilon C_1+\epsilon^2C_2+\ldots.\\ h&=h_0+\epsilon h_1+\epsilon^2h_2+\ldots.\\ \mu_3&=\mu_{3_0}+\epsilon\mu_{3_1}+\epsilon^2\mu_{3_2}+\ldots.\\ \mu_4&=\mu_{4_0}+\epsilon\mu_{4_1}+\epsilon^2\mu_{4_2}+\ldots.\\ S&=S_0+\epsilon S_1+\epsilon^2S_2+\ldots.\end{aligned} \tag{33}$$

substitute the expansions into expressions (25)–(32) and equate coefficients of like powers in ϵ. The resulting boundary-value problems are used to determine the unknown functions. Thus, the first approximation to the stress function is the solution of the following boundary-value problem

$$\phi_{0_{tt}}=0,\quad \phi_0(s,h_0)=0,\quad \phi_0(s,0)=C_0. \tag{33}$$

The unknown constant C_0 is found from

$$\int_0^1 \phi_{0_t}(s,0)\,\mathrm{d}s=-2\Omega, \tag{34}$$

and the Lagrange multipliers, μ_{3_0} and μ_{4_0} appearing in the optimality condition,

$$\phi_{0_t}(s,h_0)=-\mu_{3_0}\sqrt{\left(1+\mu_{4_0}\sin^2\frac{s}{R}\right)} \tag{35}$$

are determined from the isoperimetric conditions

$$2C_0\Omega=J^0,\quad R^2\int_0^1 h_0\sin^2\left(\frac{\mathrm{s}}{\mathrm{R}}\right)\mathrm{d}s=I^0. \tag{36}$$

Similarly the second approximation to the stress function is the solution of the boundary-value problem

$$\phi_{1_{tt}} + \frac{1}{R}\phi_{0_t} = -2$$
$$\phi_1(s, h_0) = 0 \tag{37}$$
$$\phi_1(s, 0) = C_1,$$

the unknown constant C_1 being given by

$$\int_0^1 \phi_{1_t}(s, 0)\,\mathrm{d}s = 0, \tag{38}$$

and the Lagrange multipliers μ_{3_1} and μ_{3_2}, appearing in the optimality condition,

$$\phi_{1_t}(s, h_0)\,\phi_{0_t}(s, h_0) = \mu_{3_0}{}^2\left[\frac{\mu_{3_1}}{\mu_{3_0}} + \left(\frac{\mu_{4_1}}{2} + \frac{\mu_{3_1}}{\mu_{3_0}}\mu_{4_0} + \mu_{4_0}\frac{h_0}{R}\right)\sin^2\frac{s}{R}\right] \tag{39}$$

are, as usual, determined from the isoperimetric conditions

$$\int_0^1\int_0^{h_0}\phi_0\,\mathrm{d}t\,\mathrm{d}s + C_1\Omega = 0, \quad \int_0^1\left(h_1 + \frac{3}{2}\frac{h_0^2}{R}\right)\sin^2\left(\frac{s}{R}\right)\mathrm{d}s = 0. \tag{40}$$

To within the terms of $0(\epsilon^2)$ the cross-sectional area of the optimal bar is given by

$$\begin{aligned} S &= S_0 + \epsilon S_1 + 0(\epsilon^2), \\ &= \int_0^1\left[h_0 + \frac{H}{L}\left(h_1 + \frac{h_0^2}{2R}\right)\right]\mathrm{d}s. \end{aligned} \tag{41}$$

We now present the first approximate solution to the optimization problem under consideration. Although it is also possible to obtain the second approximation to the solution in an analytical form, in our opinion the effort involved is hardly worth the improvement achieved. Moreover, the first approximation itself reveals the general shape of the optimal cross-section and leads to a substantial economy in area (weight).

First approximation to the solution:

From the differential equation, boundary conditions (33) and the optimality condition (35) it follows that

$$\phi_0 = C_0\left(1 - \frac{t}{h_0}\right), \quad h_0 = C_0/\mu_{3_0}\sqrt{\left(1 + \mu_{4_0}\sin^2\frac{s}{R}\right)}. \tag{42}$$

The unknowns C_0, μ_{3_0} and μ_{4_0} are determined from (34) and (36). From the first isoperimetric condition (36) it directly follows that

$$C_0 = J^0/2\Omega, \tag{43}$$

while (34) gives

$$\begin{aligned} \mu_{3_0} &= \pi\Omega/E(\sqrt{(-\mu_{4_0})}; -1 < \mu_{4_0} \le 0. \\ &= \pi\Omega/\sqrt{(1 + \mu_{4_0})}\,E[\sqrt{(\mu_{4_0}/1 + \mu_{4_0})}];\ \mu_{4_0} \ge 0 \end{aligned} \tag{44}$$

where $E(\ )$ is the complete elliptic integral of the second kind.

Finally, the second isoperimetric condition (36) leads to

$$\begin{aligned} I^0 &= \frac{4R^3C_0}{\mu_{3_0}\mu_{4_0}}[E[\sqrt{(-\mu_{4_0})}] - F[\sqrt{(-\mu_{4_0})}]]; \quad -1 < \mu_{4_0} \le 0 \\ &= \frac{4R^3C_0}{\mu_{3_0}\mu_{4_0}}\left[\sqrt{(1 + \mu_{4_0})}\,E[\sqrt{(\mu_{4_0}/1 + \mu_{4_0})}] - \frac{F[\sqrt{(\mu_{4_0}/1 + \mu_{4_0})}]}{\sqrt{(1 + \mu_{4_0})}}\right]; \quad \mu_{4_0} \ge 0 \end{aligned} \tag{45}$$

where $F(\ )$ is the complete elliptic integral of the first kind.

Note that in the non-dimensional form $\Omega = 1/4\pi$ and $R = 1/2\pi$. Taking this into account and also substituting (43) and (44) into (45), we get

$$\begin{aligned}\frac{I^0}{J^0} &= \frac{4}{\pi^2}\frac{E[\sqrt{(-\mu_{4_0})}]}{\mu_{4_0}}[E[\sqrt{(-\mu_{4_0})}] - F[\sqrt{(-\mu_{4_0})}]]; \quad -1 < \mu_{4_0} \le 0 \\ &= \frac{4}{\pi^2}\frac{E[\sqrt{(\mu_{4_0}/1+\mu_{4_0})}]}{\mu_{4_0}}[(1+\mu_{4_0})E[\sqrt{(\mu_{4_0}/1+\mu_{4_0})}] - F[\sqrt{(\mu_{4_0}/1+\mu_{4_0})}]]; \quad \mu_{4_0} \ge 0. \end{aligned} \tag{46}$$

The relation between I^0/J^0 and μ_{4_0} is represented graphically in Fig. 3.

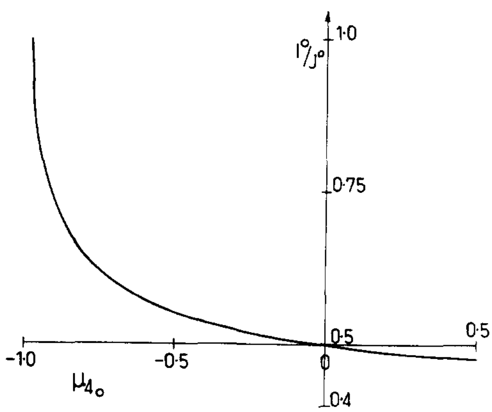

Fig. 3

By expanding the elliptic integrals in powers of their arguments, it is easy to show that, as $\mu_{4_0} \to 0$, $I^0/J^0 \to 1/2$ which corresponds to the line dividing the regions 1 and 2 in the plane of design parameters (J, I) (Fig. 2). Indeed, as $\mu_{4_0} \to \infty$, $I^0/J^0 \to 0$.

It follows, therefore, that when $\mu_{4_0} \ge 0$ the area of cross-section of the "optimal" bar would be greater than that of the corresponding hollow circular bar. This is indeed so, as is evident from the expression for the cross-sectional area of the optimal bar

$$\begin{aligned} S &= \frac{2RJ^0}{\pi\Omega^2}E[\sqrt{(-\mu_{4_0})}]\,F[\sqrt{(-\mu_{4_0})}]; \quad -1 < \mu_{4_0} \le 0 \\ &= \frac{2RJ^0}{\pi\Omega^2}E\left[\sqrt{\left(\frac{\mu_{4_0}}{1+\mu_{4_0}}\right)}\right]F\left[\sqrt{\left(\frac{\mu_{4_0}}{1+\mu_{4_0}}\right)}\right]; \quad \mu_{4_0} \ge 0. \end{aligned} \tag{47}$$

To get an idea of the economy achieved even in the first approximation, we compare the cross-sectional area of the optimal bar with a thin-walled circular cross-section having the same stiffness. In region 1 (Fig. 2) a thin-walled circular cross-section is the optimal solution and hence the ratio S/S_{cir} will be greater than unity, or, which is the same as, $(S - S_{cir}/S_{cir}) > 0$. Note, however, that S_{cir} in region 1 is governed only by the torsional stiffness, because a circular cross-section with torsional stiffness J^0 will necessarily have a bending stiffness greater than I^0. The converse is true in region 2.

In region 1, where $\mu_{4_0} > 0$

$$S_{cir} = 4\pi^2 J^0,$$

and

$$S/S_{cir} = \frac{4}{\pi^2}E\left[\sqrt{\left(\frac{\mu_{4_0}}{1+\mu_{4_0}}\right)}\right]F\left[\sqrt{\left(\frac{\mu_{4_0}}{1+\mu_{4_0}}\right)}\right]. \tag{48}$$

In region 2, where $\mu_{4_0} < 0$

$$S_{cir} = 8\pi^2 I^0,$$

and

$$S/S_{\text{cir}} = \frac{\mu_{4_0}}{2} \frac{F[\sqrt{(-\mu_{4_0})}]}{[E[\sqrt{(-\mu_{4_0})}] - F[\sqrt{(-\mu_{4_0})}]]}. \tag{49}$$

The expressions (48) and (49) are shown graphically in Fig. 4.

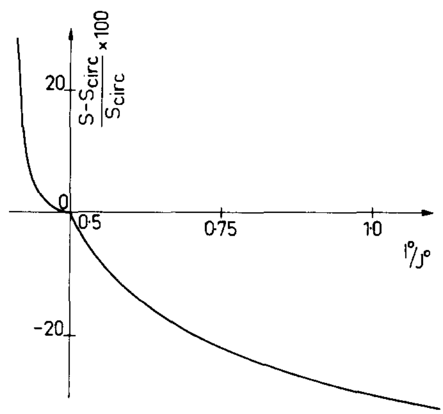

Fig. 4

It is clear that substantial economy is possible for $I^0/J^0 > 1/2$. It should however be emphasized that the results for large values of I^0/J^0 are not realiable in view of the assumption regarding the wall thickness of the hollow cross-section. Nevertheless, substantial economy is clearly possible even for those values of I^0/J^0 for which the hollow cross-section can be considered as thin-walled.

Given I^0 and J^0, the variation in the non-dimensional wall thickness, h_0, is calculated from

$$h_0/J^0 = 8\pi E[\sqrt{(-\mu_{4_0})}]/\sqrt{(1 + \mu_{4_0}\sin^2\theta)}, \mu_{4_0} < 0 \tag{50}$$

where the value of μ_{4_0} corresponding to the given design constraints is found from Fig. 3, and θ varies between 0 and 2π.

The cross-sectional shape in region 2 for a typical value of I^0/J^0 is shown in Fig. 2. The thickness variation of the optimal cross-section is evident even in the first approximation. For the corresponding problem with a single constraint on the torsional stiffness only, the wall thickness is constant in the first approximation[2] which necessitates a refinement of the solution.

3. CONCLUSIONS

The present work considered minimum-weight (area) design of cylindrical bars with a doubly-connected (hollow) cross-section used to withstand either torsional or bending moments at any given time in their design life. The design was to be such that, given the shape of the inner boundary, the cross-section had to have certain prescribed torsional and bending stiffnesses. In other words, the shape of the outer boundary had to be determined. The necessary optimality condition was derived by including the constraints on stiffnesses through the use of Lagrange multipliers. It was shown that the optimality condition is identical to that for a corresponding bar with a solid cross-section.

The boundary-value problem together with the optimality condition was solved by a perturbation technique under the assumption that the wall thickness of the hollow cross-section was small in comparison with the length of the inner contour, whose shape was known. It was shown that, already in the first approximation, the shape of the optimal cross-section is more or less revealed, and that substantial economy is possible if the ratio of the prescribed bending to torsional stiffness exceeds one-half. It was further shown that, when this ratio is less than one-half and the inner boundary is circular in shape, the outer boundary will also be circular.

Finally, it should be noted that the shape of the optimal cross-section could be slightly different if more terms were included in the solution. However, in our opinion, the refinement so achieved would not be worth the effort spent on solving the required boundary-value problems.

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